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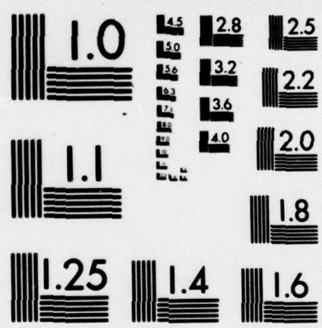

MICROCOPY RESOLUTION TEST CHART

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 79-0602	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) HOPF BIFURCATION FOR FUNCTIONAL EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jack K. Hale and Jose Carlos F. De Oliveira		8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3092; DAAG27-76-G-0294
9. PERFORMING ORGANIZATION NAME AND ADDRESS Brown University, Division of Applied Mathematics, Providence, Rhode Island 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332		12. REPORT DATE January 1979
		13. NUMBER OF PAGES 32
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

LEVEL II

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

37 p.

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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AFOSR-TR- 79-0602

HOPF BIFURCATION FOR FUNCTIONAL EQUATIONS

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+This research was supported in part by the Air Force Office of Scientific Research under AF-AFOSR 77-3092A, in part by the National Science Foundation under MCS 76-07246-03 and in part by the U.S. Army Research Office under ARO-DAAG 27-76-G-0294.

++This research was supported in part by FAPESP-Fundação de Amparo à Pesquisa do Estado de São Paulo, under proc. 77/0741.

HOPF BIFURCATION FOR FUNCTIONAL EQUATIONS

Introduction. The purpose of this paper is to study the existence of a smooth Hopf bifurcation for functional equations. The bifurcation parameters may include the delays. The results will be described for a special case of the equations considered.

Suppose r_1, r_2, r_3 are given positive numbers, $a(\theta)$, $-r \leq \theta \leq 0$ is a C^1-function, $g: \mathbb{R}^3 \to \mathbb{R}$ is a C^1-function, $g(0) = 0$, $h: \mathbb{R} \to \mathbb{R}$ is a C^1-function, $h(0) = 0$, and consider the equation

$$(1) \qquad x(t) - g\left(x(t-r_1),\ x(t-r_2),\ \int_{-r_3}^{0} a(\theta)h(x(t+\theta))d\theta\right) = 0.$$

Suppose the linear variational equation around zero,

$$(2) \qquad x(t) - a_1 x(t-r_1) - a_2 x(t-r_2) - a_3 \int_{-r_3}^{0} a(\theta)x(t+\theta)d\theta = 0,$$

has the property that there is a point $\alpha^0 = (a_1^0, a_2^0, a_3^0, r_1^0, r_2^0) \in \mathbb{R}^3 \times (\mathbb{R}_+)^2$ and a surface S through this point of codimension one such that the characteristic equation

$$(3) \qquad 1 - a_1 e^{-\lambda r_1} - a_2 e^{-\lambda r_3} - a_3 \int_{-r}^{0} a(\theta) e^{\lambda\theta} d\theta = 0$$

has two roots $\lambda(\alpha), \overline{\lambda}(\alpha)$, $\alpha = (a_1, a_2, a_3, r_1, r_2)$, for α in a neighborhood U of S, $\lambda(\alpha^0) = i\nu$, $\nu > 0$, and the remaining roots are bounded away from the imaginary axis for $\alpha \in U$. Also, suppose $d\lambda(\alpha)/d\alpha \neq 0$ along the normal to the tangent plane of S

for $\alpha \in U$. Under this hypothesis, it is shown that Equation (1) has a smooth Hopf bifurcation from 0 at any point on S.

One important remark is that the delays r_1, r_2 can be chosen as bifurcation parameters. At first, it would seem to be impossible to prove this result since the function in Equation (1) considered as a function on the space $C([-h,0],\mathbb{R})$, $h > \max(r_1, r_2, r)$ is not a differentiable function in r_1, r_2. However, under the assumption that $a(\theta)$ is C^1 in θ, we prove every periodic solution of Equation (1) must be C^1. This fact and an argument similar in spirit to the one in [4] make it possible to prove the bifurcation theorem.

Our proof of the theorem uses only the Fourier series of a periodic function and not the variation of constants formula as is usually the case for evolutionary equations (see, for example, [3]).

To determine the number of periodic solutions that bifurcate from zero at α and their stability properties, one must use some type of averaging process along with the variation of constants formula. This latter formula is used either to obtain a center manifold theorem or to show that the characteristic multipliers of the linear variational equation around a periodic orbit determine the asymptotic behavior of the solutions. These topics will be treated in a subsequent paper.

The results hold for more general equations

$$x(t) - \sum A_k x(t-r_k) - \int_{-r}^{0} A(\theta)x(t+\theta)d\theta - g(\alpha, x_t) = 0$$

where $x \in \mathbb{R}^n$, $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$, and α is the bifurcation parameter which may include all the r_k and coefficient matrices $A_k, A(\theta)$.

1. Functional Equations.

Suppose $r \geq 0$ is a given real number, $\mathbb{R}$ is the real line, $\mathbb{R}^n$ is an n-dimensional normed space over $\mathbb{R}$ with norm $|\cdot|$, $C([a,b],\mathbb{R}^n)$ is the Banach space of all continuous functions $\phi: [a,b] \to \mathbb{R}^n$ with the norm $|\phi| = \sup\{|\phi(\theta)|: a \leq \theta \leq b\}$, $C = C([-r,0],\mathbb{R}^n)$, $\mathscr{L}(X,Y)$ is the space of all linear continuous mappings $L: X \to Y$ from the Banach space $(X,|\cdot|)$ into the Banach space $(Y,|\cdot|)$, with the norm $|L| = \sup\{|L\phi|: |\phi| = 1\}$, $\mathscr{L}(X,X) = \mathscr{L}(X)$. If $x \in C([a,b],\mathbb{R}^n)$ and $a + r \leq t \leq b$ we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

Recall the Riesz Theorem [8] on the representation of elements of $\mathscr{L}(C,\mathbb{R}^n)$ via the Stieltjes Integral: any $D \in \mathscr{L}(C,\mathbb{R}^n)$ can be written in the form

$$D\phi = A_0\phi(0) - \int_{-r}^{0} d\mu(\theta)\phi(\theta), \quad \phi \in C,$$

where $A_0 \in \mathscr{L}(\mathbb{R}^n)$ and $\mu: [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ is a function of bounded variation on $[-r,0]$, continuous at $\theta = 0$.

If we decompose μ as the sum of its saltus part, plus its absolutely continuous part, plus its singular part, we can also write $D\phi$ as

$$D\phi = A_0\phi(0) - \sum_k A_k\phi(-r_k) - \int_{-r}^{0} A(\theta)\phi(\theta)d\theta - \int_{-r}^{0} [dS(\theta)]\phi(\theta),$$

where $\{-r_k\}_k$ is the countable set of discontinuities of μ, $A_k \in \mathscr{L}(\mathbb{R}^n)$ is the jump of μ at $-r_k$, for all k,

$\sum_k |A_k| < \infty$, $\lim_{\varepsilon \downarrow 0} \sum_{0<r_k<\varepsilon} |A_k| = 0$, $A: [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ is absolutely Lebesgue integrable i.e., $A \in L^1$ and $S: [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ is continuous, has bounded variation on $[-r,0]$ and $\frac{dS}{d\theta} = 0$ almost everywhere.

In what follows, we always suppose that the singular part S is identically zero. Define

$$D^0\phi = A_0\phi(0) - \sum_k A_k\phi(-r_k);$$

$$D^1\phi = -\int_{-r}^{0} A(\theta)\phi(\theta)d\theta, \quad \text{for all } \phi \in C,$$

and $H^0(\lambda) = D^0(e^{\lambda\cdot}I) = A_0 - \sum_k A_k e^{-\lambda r_k}$ and $H(\lambda) = D(e^{\lambda\cdot}I)$, for all $\lambda \in \mathbb{C}$, where I is the identity $n \times n$ matrix.

We say that D^0 is <u>hyperbolic</u> if there exist constants $a > 0$ and $b > 0$ such that $|\det H^0(\lambda)| \geq b$ for all λ such that $|\mathrm{Re}\,\lambda| \leq a$. If $|\det H^0(\lambda)| \geq b$ for all λ such that $\mathrm{Re}\,\lambda \geq -a$, we say that D^0 is <u>stable</u>. To justify this terminology we note that if A_0 is invertible in $\mathscr{L}(\mathbb{R}^n)$, the hyperbolic (resp. stable) functionals D^0 are characterized by the fact that the origin $0 \in C^0 =_{def} \{\phi \in C: D^0\phi = 0\}$ is a hyperbolic (resp. stable) equilibrium point of the linear dynamical system defined on C^0 by $T^0(t)\phi = x_t$, for all $t \geq 0$ and $\phi \in C^0$, where $x: [-r,\infty) \to \mathbb{R}^n$ is the continuous solution of the initial value problem $D^0(x_t) = 0$, $x_0 = \phi$. See [3] and [5] for details. The definition of hyperbolicity and stability can be extended to a general $D \in \mathscr{L}(C,\mathbb{R}^n)$ and the above characterization extends to the case when the singular part S is identically zero; the case $S \not\equiv 0$ is still open. This concept of hyperbolic is a special case of admissibility in [6], and appeared implicitly in [5].

Using the Riemann-Lebesgue Lemma, it is easy to prove that, if D^0 is hyperbolic, then the function $\det H(\lambda)$ can have only a finite number of zeros in a sufficiently thin neighborhood of the imaginary axis. Of course, all of them have finite multiplicity. Therefore, if D^0 is hyperbolic, then $D = D^0 + D^1$ is hyperbolic if and only if the function $\det H(\lambda)$ does not have purely imaginary roots.

In what follows, we always suppose that D^0 is hyperbolic.

Consider the functional equation

$$D(x_t) - g(x_t) = f(t) \tag{1.1}$$

where $g\colon C \to \mathbb{R}^n$ is continuously differentiable, $g(0) = 0$, $g'(0) = 0$ and $f\colon \mathbb{R} \to \mathbb{R}^n$ is continuous.

We study the periodic solutions of Equation (1.1) when f is periodic. Let $\mathcal{P}_\omega$, $\omega > 0$, be the space of all continuous ω-periodic functions $f\colon \mathbb{R} \to \mathbb{R}^n$ with the norm $|f| = \sup\{|f(t)|: t \in \mathbb{R}\}$, let $\mathcal{P}_\omega^{(1)} = \{f \in \mathcal{P}_\omega: \frac{df}{dt} \in \mathcal{P}_\omega\}$ and for each integer n, let $c_n[f] = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt$.

<u>Lemma 1.1</u>. If D^0 is hyperbolic, then for any $\omega > 0$ and for any $f \in \mathcal{P}_\omega$, there exists a unique solution of $D^0 x_t = f(t)$ in $\mathcal{P}_\omega$.

If we denote this solution by $x = Sf$, then S is a bicontinuous bijective operator from $\mathcal{P}_\omega$ onto itself. If f is C^k, so is Sf.

Proof. For simplicity, let us suppose $\omega = 2\pi$. If $f \in \mathcal{P}_{2\pi}$ and if the equation $D^0 x_t = f(t)$ has a solution x in $\mathcal{P}_{2\pi}$, then

$$c_n[x] = [H^0(in)]^{-1} c_n[f] \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots .$$

This shows uniqueness of the solution in $\mathcal{P}_{2\pi}$ (and existence and uniqueness of solution in $L^2[0,2\pi]$). To prove existence of solution in $\mathcal{P}_{2\pi}$ we use a result of Cameron and Pitt [7] which states that there exist sequences $\{X_k\}$, $\{\gamma_k\}$, $X_k \in \mathcal{L}(\mathbb{R}^n)$, $\gamma_k \in \mathbb{R}$, $\sum_k |X_k| < \infty$, such that $[H^0(in)]^{-1} = \sum_k X_k e^{-in\gamma_k}$ for all $n \in \mathbb{R}$. It is easy to see now that $x(t) = \sum_k X_k f(t+\gamma_k)$ is a continuous 2π-periodic solution of $D^0 x_t = f(t)$ and the lemma is apparent.

As an example, consider the scalar equation

$$x(t) - A_1 x(t-1) - A_2 x(t-\pi) = \text{cost}$$

under the hypothesis $|A_1| + |A_2| < 1$.

The functional $D^0\phi = \phi(0) - A_1\phi(-1) - A_2\phi(-\pi)$ is stable and

$$\frac{1}{H^0(in)} = \sum_{k=0}^{\infty} (A_1 e^{-in} + A_2 e^{-in\pi})^k$$

and the unique 2π-periodic solution of the above equation is given by

$$x(t) = \sum_{k=0}^{\infty} (-1)^k \sum_{p=0}^{\infty} (-1)^p A_1^p A_2^{k-p} \cos(t+p).$$

Even if D^0 is not hyperbolic, we can find, by the above procedure the ω-periodic solutions of $D^0(x_t) = f(t)$, $f \in \mathscr{P}_\omega$, in the case where f is a trigonometric polynomial. But the general situation is very complicated. For the example $x(t) + \frac{1}{2} x(t-1) + \frac{1}{2} x(t-\pi) = f(t)$, with $f \in \mathscr{P}_{2\pi}$ arbitrary, one encounters the problem of small divisors.

Another application of Lemma 1.1 is the equation

$$x(t-1) + \sum_{k=2}^{\infty} \frac{1}{2^{k+1}} [x(t - \tfrac{1}{k}) + x(t - 2 + \tfrac{1}{k})] = f(t).$$

In this example,

$$D^0(\phi) = \phi(-1) + \sum_{k=2}^{\infty} \frac{1}{2^{k+1}} [\phi(- \tfrac{1}{k}) + \phi(-2 + \tfrac{1}{k})]$$

and it is not difficult to prove that D^0 is hyperbolic. Therefore, the above equation has a unique solution in $\mathscr{P}_\omega$ for any $f \in \mathscr{P}_\omega$.

<u>Lemma 1.2</u>. If D^0 is hyperbolic and $f \in \mathscr{P}_\omega$, then, there exists a $\delta = \delta(D^0) > 0$ such that, for any continuous $T: \mathbb{R} \to \mathscr{L}(C, \mathbb{R}^n)$ such that $T(t+\omega) = T(t)$ for all $t \in \mathbb{R}$ and $|T| < \delta$, where $|T| =_{def} \sup\{|T(t)\phi| : \phi \in C, |\phi| = 1, t \in \mathbb{R}\}$, the equation

$$D^0(x_t) - T(t)x_t = f(t)$$

has a unique solution $x(T,f) \varepsilon \mathscr{P}_\omega$ and the map $(T,f) \mapsto x(T,f)$ is continuous.

Proof. Let $\mathscr{L}$ be the space of all maps $T: \mathbb{R} \to \mathscr{L}(C,\mathbb{R}^n)$ which are continuous and ω-periodic, with the above norm $|T|$. Consider the mapping

$$\mathscr{F}: \mathscr{L} \times \mathscr{P}_\omega \to \mathscr{P}_\omega$$

defined by $\mathscr{F}(T,x)(t) = D^0(x_t) - T(t)x_t - f(t)$ for all $t \varepsilon \mathbb{R}$. It is easy to see that $\mathscr{F}$ is continuously differentiable. By Lemma 1.1, $\mathscr{F}(0,x_f) = 0$ where $x_f(t) = \sum_k X_k f(t+\gamma_k)$. Lemma 1.1 again implies that $\frac{\partial \mathscr{F}}{\partial x}(0,x_f)$ is an isomorphism from $\mathscr{P}_\omega$ onto itself.

Thus, the Implicit Function Theorem (IFT) implies the conclusions in Lemma 1.2.

Lemma 1.3. If D^0 is hyperbolic and $f \varepsilon \mathscr{P}_\omega$, then there exists a $\delta = \delta(D^0) > 0$ such that, for any continuously differentiable map $g: C \to \mathbb{R}^n$ such that $|g| < \delta$, where $|g| = \sup\{|g^{(j)}(\phi)|: \phi \varepsilon C, j = 0,1\}$, then the equation

$$D^0(x_t) - g(x_t) = f(t)$$

has a unique solution $x \varepsilon \mathscr{P}_\omega$. Moreover, if f is C^1, so is x.

Proof. Let $\mathscr{X} = \{g: C \to \mathbb{R}^n \mid g$ is C^1, $|g| < \infty\}$ and consider the mapping

$$\mathscr{F}: \mathscr{X} \times \mathscr{P}_\omega \to \mathscr{P}_\omega$$

defined by

$$\mathscr{F}(g, x)(t) = D^0(x_t) - g(x_t) - f(t)$$

for all $t \in \mathbb{R}$.

By Lemma 1.1, $\mathscr{F}(0, \sum_k X_k f(\cdot+\gamma_k)) = 0$ and $\frac{\partial \mathscr{F}}{\partial x}(0, \sum X_k f(\cdot+\gamma_k))$ is an isomorphism from $\mathscr{P}_\omega$ onto $\mathscr{P}_\omega$.

The IFT can be applied to give a solution $x = x(g,f)$ in $\mathscr{P}_\omega$ of our equation.

We suppose now that f is C^1 and prove that $x(g,f)(t)$ is continuously differentiable in $t \in \mathbb{R}$. Let $y \in \mathscr{P}_\omega$ be the unique solution of

$$D^0(y_t) - g'(x_t(g,f)) \cdot y_t = \dot{f}(t).$$

Consider the function

$$z(t,\Delta t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} - y(t)$$

defined for $t \in \mathbb{R}$, $\Delta t \neq 0$. As a function of t, $z(t,\Delta t)$ satisfies the equation

$$D^0(z_t) - [\int_0^1 g'(\zeta x_{t+\Delta t} + (1-\zeta)x_t)d\zeta]z_t = \int_0^1 [\dot{f}(t+\zeta\Delta t) - \dot{f}(t)]d\zeta$$

$$+ (\int_0^1 [g'(\zeta x_{t+\Delta t} + (1-\zeta)x_t) - g'(x_t)]d\zeta)y_t .$$

Since this equation depends continuously upon Δt, for $t \in \mathbb{R}$. Lemma 1.2 implies that $z(\cdot,\Delta t)$ is a continuous function of Δt, for $\Delta t \in \mathbb{R}$. Therefore, there exists the limit $\lim_{\Delta t \to 0} z(\cdot,\Delta t) \in \mathscr{P}_\omega$. But this limit solves the limit equation $D^0(z_t) - g'(x_t) \cdot z_t = 0$, so it is the zero function and $\dot{x} = y$.

<u>Corollary 1.4</u>. Suppose D^0 is hyperbolic, f is C^1 and $A: [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ is an L^1-function such that the function $t \mapsto \int_{-r}^0 A(\theta)x(t+\theta)d\theta$ is continuously differentiable whenever x is continuous. Then, there exists a $\delta = \delta(D^0) > 0$ such that if $g: C \to \mathbb{R}^n$ is C^1, and $|g| < \delta$, then, all continuous periodic solutions of the equation

$$D^0(x_t) - \int_{-r}^0 A(\theta)x(t+\theta)d\theta - g(x_t) = f(t)$$

are C^1.

<u>Proof</u>. Suppose $x^* \in \mathscr{P}_\omega$ is a solution of the above equation. Then, x^* is also a solution of

$$D^0(x_t) - g(x_t) = \int_{-r}^0 A(\theta)x^*(t+\theta)d\theta + f(t)$$

and Lemma 1.3 implies the result.

Suppose D^0 is hyperbolic and suppose that D is not hyperbolic. Let us seek now the solutions in $\mathscr{P}_\omega$ of the linear homogeneous functional equation $D(x_t) = 0$. We consider only the case $\omega = 2\pi$. By taking Fourier coefficients, we see that there exist nonzero solutions in $\mathscr{P}_{2\pi}$ if and only if the equation $\det H(in) = 0$ admits at least one integer solution n. Let $n_1,\dots,n_m$ be all these integers. Let $v_{n_j}^{(1)},\dots,v_{n_j}^{(p_j)}$ be a basis for the subspace $\{v \in \mathbb{R}^n : H(in_j)v = 0\}$. Then, it is easy to prove that the set of all complex-valued continuous 2π-periodic solutions of $Dx_t = 0$ is the set of all linear combinations of the functions $e^{in_jt}v_{n_j}^{(k)}$; $k = 1,\dots,p_j$; $j = 1,\dots,m$.

We now study the periodic solutions of the nonhomogeneous equation $D(x_t) = f(t)$, when $f \in \mathscr{P}_\omega$, D^0 is hyperbolic but $D = D^0 + D^1$ is not hyperbolic.

For this purpose, we define the adjoint equation associated to the equation $D(x_t) = 0$ as the functional equation

$$D^*(y_t) \equiv y(t)A_0 - \sum_k y(t+r_k)A_k - \int_{-r}^{0} y(t-\theta)A(\theta)d\theta = 0$$

where y belongs to the dual $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$. All that has been said for the equation $D(x_t) = 0$ carries over for Equation (1.5) with the obvious adaptation. In particular, the set of all continuous 2π-periodic solutions of $D^*(y_t) = 0$ is the set of all linear combinations of the functions $e^{in_jt}w_{n_j}^{(k)}$; $k = 1,\dots,p_j$;

$j = 1,\ldots,m$ where $w_{n_j}^{(1)},\ldots,w_{n_j}^{(p_j)}$ is a basis for the subspace $\{w \in (\mathbb{R}^n)^*: wH(in_j) = 0\}$ and as before, $n_1,\ldots,n_j$ are the integer solutions of $\det H(in) = 0$. We will suppose that the vectors $w_{n_j}^{(k)}$ are unit vectors.

<u>Lemma 1.5</u> (Fredholm Alternative) Suppose D^0 is hyperbolic, $D = D^0 + D^1$ is not hyperbolic, $A: [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ belongs to L^1 and $f \in \mathscr{P}_\omega$. Then the equation $Dx_t = f(t)$ has a solution in $\mathscr{P}_\omega$ if and only if f is orthogonal to the continuous ω-periodic solutions of the homogeneous adjoint equation $D^*y_t = 0$ that is, $\int_0^\omega y(t)f(t)dt = 0$ for all continuous ω-periodic functions $y: \mathbb{R} \to (\mathbb{R}^n)^*$ such that $D^*(y_t) = 0$. Moreover, if f is C^1 and $t \mapsto \int_{-r}^0 A(\theta)x(t+\theta)d\theta$ is C^1 whenever x is continuous, then, all solutions of $D(x_t) = f(t)$ in $\mathscr{P}_\omega$ are in $\mathscr{P}_\omega^{(1)}$.

Furthermore, there exist linear continuous operators $J: \mathscr{P}_\omega \to \mathscr{P}_\omega$ and $\mathscr{K}: (I-J)\mathscr{P}_\omega \to \mathscr{P}_\omega$ such that $(I-J)\mathscr{P}_\omega$ is the set of all $f \in \mathscr{P}_\omega$ which satisfies the above orthogonality condition and, for any $f \in (I-J)\mathscr{P}_\omega$, $x = \mathscr{K}f$ is a solution in $\mathscr{P}_\omega$ of the equation $D(x_t) = f(t)$. In other words, $\mathscr{K}$ is a continuous right inverse of the operator $x \mapsto D(x_\bullet)$.

<u>Proof</u>. For simplicity take $\omega = 2\pi$. If the equation has a solution $x^*(t)$ in $\mathscr{P}_{2\pi}$, then equating Fourier coefficients

implies $H(in_j)c_{n_j}[x^*] = c_{n_j}[f]$.

Multiplying both sides of this equality by $w_{n_j}^{(k)}$ we get

$$w_{n_j}^{(k)} \cdot c_{n_j}[f] = 0$$

or

$$\int_0^{2\pi} e^{-in_jt}\, w_{n_j}^{(k)} \cdot f(t) = 0$$

for all $k = 1,\ldots,p_j$; $j = 1,\ldots,m$. But this implies that f is orthogonal to 2π-periodic solutions of $D^*y_t = 0$.

Conversely, if the above integrals are zero, then we can produce vectors $v_f^{(n_j)}$, $j = 1,\ldots,m$, such that $H(in_j)v_f^{(n_j)} = c_{n_j}[f]$ and construct the function

$$x^*(t) = \sum_{j=1}^{m} e^{in_jt} \cdot v_f^{(n_j)} + \sum_{n\neq n_1,\ldots,n_m} H^{-1}(in)\cdot c_n[f]e^{int}.$$

This function is a well-defined 2π-periodic L^2-solution of the equation $D(x_t) = f(t)$. From the Schwarz Inequality, x^* is bounded and therefore x^* belongs to $L^\infty[0,2\pi]$. Let us prove that $x^*(t)$ is in fact a continuous function. Indeed, $x^*(t)$ is also a 2π-periodic L^2-solution of the equation

$$D^0(x_t) = f(t) + \int_{-r}^{0} A(\theta)x^*(t+\theta)d\theta.$$

By hypothesis, f is continuous; the function $t \mapsto \int_{-r}^{0} A(\theta)x(t+\theta)d\theta$ is continuous since it is the convolution of an L^1-function with an

L^∞ function. Then, Lemma 1.1 implies that x^* is continuous.

Now, since the matrix $H(in_j)$ admits a right inverse, the vectors $v_f^{(n_j)}$ can be chosen such that they are linear and continuous in f.

We define the projection J by

$$(Jf)(t) = \sum_{j=1}^{m} \sum_{k=1}^{p_j} e^{in_j t}\left[\frac{1}{2\pi}\int_0^{2\pi} w_{n_j}^{(k)} e^{-in_j s} f(s)ds\right]\left[\bar{w}_{n_j}^{(k)}\right]^T, \quad t \in \mathbb{R},\ f \in \mathscr{P}_\omega$$

and the operator $\mathscr{K}\colon (I-J)\,\mathscr{P}_\omega \to \mathscr{P}_\omega$ by

$$(\mathscr{K}f)(t) = \sum_{j=1}^{m} e^{in_j t} v_f^{(n_j)} + \sum_{n \neq n_1,\dots,n_m} H^{-1}(in)c_n[f]e^{int}, \quad t \in \mathbb{R},$$

$$f \in (I-J)\,\mathscr{P}_\omega .$$

Schwarz inequality shows that $\mathscr{K}$ is continuous and the rest of the proof of Lemma 1.5 is obvious.

Let us specialize the Fredholm alternative to the case where $\pm i$ are simple roots of $h(\lambda) = \det H(\lambda) = 0$, that is, $h(i) = 0$ and $h'(i) \neq 0$. Let P be an invertible $n \times n$ matrix such that $P^{-1}H(i)P$ is in the complex Jordan canonical form

$$P^{-1}H(i)P = \begin{bmatrix} \mu_1 & \varepsilon_1 & & & \bigcirc \\ & \mu_2 & \varepsilon_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varepsilon_{n-1} \\ \bigcirc & & & & \mu_n \end{bmatrix}$$

where $\mu_1,\ldots,\mu_n$ are the eigenvalues of $H(i)$ and $\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}$. Since 0 is an eigenvalue of $H(i)$ we can suppose that $\mu_1 = 0$. Let $M(\lambda) = [M_{ij}(\lambda)] = P^{-1}H(\lambda)P$. Then,

$$h'(i) = \det \begin{bmatrix} M_{11}'(i) & \varepsilon_1 & 0 & \ldots & 0 \\ M_{21}'(i) & \mu_2 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & & \vdots \\ & & & & \varepsilon_{n-1} \\ M_{n1}'(i) & 0 & 0 & \ldots & \mu_n \end{bmatrix} .$$

Since $h'(i) \neq 0$ by hypothesis, we can conclude that the number of blocks associated with the eigenvalue 0 is one; that is, the null spaces $N_r = \{v \in \mathbb{C}^n : H(i)v = 0\}$ and $N_\ell = \{w \in (\mathbb{C}^n)^* : wH(i) = 0$ have dimension one. If the block corresponding to the eigenvalue 0 has dimension $j \times j$, then, it is easy to see that $v_0 = Pe_1^T$ and $w_0 = e_jP^{-1}$, where e_k is the $1 \times n$ matrix with zero components except the k^{th} which is one, are basis for N_r and N_ℓ, respectively. It is easily seen now that

$$h'(i) \begin{cases} (-1)^{j+1}M_{j1}'(i)\mu_{j+1} \cdots \mu_n & \text{if } j < n \\ (-1)^{n+1}M_{n1}(i) & \text{if } j = n \end{cases}$$

and $w_0H'(i)v_0 = M_{j1}'(i)$.

Therefore, if we define $w = \dfrac{w_0}{|w_0|}$ and $v = \dfrac{|w_0|}{M_{j1}'(i)} v_0$ we have

(1.3) $H(i)v = 0$, $wH(i) = 0$, $|w| = 1$, $wH'(i)v = 1$.

Using (1.3), one can also prove that the conditions $h(i) = 0$ and $h'(i) \neq 0$ are equivalent to saying that the equation $D(x_t) = 0$ has a one-dimensional space of solutions of the form $e^{it}b$ and no solution of the form $e^{it}[tb+c]$, $b \neq 0$.

Suppose now that $\pm i$ are simple roots of $h(\lambda) = 0$ and no other characteristic root is an integral multiple of i.

For each $f \in \mathscr{P}_{2\pi}$ such that $wc_1[f] = 0$, we choose $v[f]$

varying continuously with f such that $H(i)v[f] = c_1[f]$. Then the general real 2π-periodic continuous solution of $D(x_t) = f(t)$ is given by

$$x(t) = ce^{it}v + \bar{c}e^{-it}\bar{v} + \sum_{|n|\neq 1} H(in)^{-1}c_n[f]e^{int}$$

$$+ e^{it}v_f + e^{-it}\bar{v}_f$$

$c \in \mathbb{C}^n$ arbitrary.

The projection $J: \mathscr{P}_{2\pi} \to \mathscr{P}_{2\pi}$ takes the form

$$(1.4) \qquad (Jf)(t) = (wc_1[f])e^{it}\bar{w}^T + (\bar{w}c_{-1}[f])e^{-it}w^T$$

and the operator $\mathscr{K}$: $(I-J)\mathscr{P}_{2\pi} \to \mathscr{P}_{2\pi}$ is given by

$$(1.5) \qquad K[f](t) = e^{it}v_f + e^{-it}\bar{v}_f + \sum_{|n|\neq 1} H^{-1}(in)c_n[f]e^{int},$$

for all $f \in (I-J)\mathscr{P}_\omega$ and all $t \in \mathbb{R}$.

2. <u>The Hopf Bifurcation Theorem</u>.

Suppose $\mathscr{A}$ is a real Banach space, the <u>parameter</u> space, $\alpha_0 \in \mathscr{A}$ is fixed, V_{α_0} is a neighborhood of α_0 in $\mathscr{A}$; D and g are continuous mappings from $\mathscr{A} \times C$ into $\mathbb{R}^n$; $D(\alpha)\phi$ is linear in ϕ,

$$D(\alpha) = D^0(\alpha) + D^1(\alpha)$$

$$D^0(\alpha)\phi \equiv A_0(\alpha)\phi(0) - \sum_k A_k(\alpha)\phi(-r_k(\alpha)),\ D^1(\alpha)\phi = -\int_{-r}^0 A(\alpha,\theta)\phi(\theta)d\theta,$$

where

$$0 < r_k(\alpha) \leq r,\ A_0(\alpha), A_k(\alpha) \in \mathscr{L}(\mathbb{R}^n),\ \sum_k |A_k(\alpha)| < \infty,\ \lim_{\varepsilon \downarrow 0} \sum_{0<r_k<\varepsilon} |A_k(\alpha)| = 0$$

uniformly in $\alpha \in V_{\alpha_0}$, $A(\alpha,\cdot)\colon [-r,0] \to \mathscr{L}(\mathbb{R}^n)$ is a L^1 function such that the function $t \mapsto \int_{t-r}^t A(\alpha,\sigma-t)x(\sigma)d\sigma$ is C^1 for any continuous $x\colon \mathbb{R} \to \mathbb{R}^n$, $g(\alpha,\phi)$ has continuous first derivative in ϕ, $g(\alpha,0) = 0$, $\frac{\partial g}{\partial \phi}(\alpha,0) = 0$, for all $\alpha \in V_{\alpha_0}$, and consider the functional equation

$$(2.1) \qquad D(\alpha)x_t - g(\alpha,x_t) = 0.$$

An example which is a special case of (2.1) is the scalar equation

$$(2.2) \qquad x(t) - A_1x(t-r_1) - A_2x(t-r_2) - A\int_{-L_2}^{-L_1} x(t+\theta)d\theta - g(x_t) = 0,$$

where the parameter α is the 7-tuple of real numbers

$$\alpha = (A_1,A_2,A,r_1,r_2,L_1,L_2).$$

Our first hypothesis concerns the behavior of the difference operator in α and is the following:

(H1) We asume there exist constants $a > 0$, $b > 0$ such that if $\lambda \in \mathbb{C}$, $|\mathrm{Re}\,\lambda| \leq a$, then

$$|\det(A_0 - \sum_k A_k e^{-\lambda r_k})| \geq b \quad \text{for all} \quad \alpha \in V_{\alpha_0}.$$

Hypothesis (H1) says that $D^0(\alpha)$ is hyperbolic for all α in a neighborhood of α_0. If α contains some of the delays r_k, then, Hypothesis (H1) puts strong restrictions on the corresponding coefficients A_k. To better understand these restrictions we reproduce here some results in [1].

Suppose

$$D^0(\alpha)\phi = \phi(0) - \sum_{k=1}^{N} A_k \phi(-\gamma_k \cdot \alpha)$$

$$\alpha = (\alpha_1, \ldots, \alpha_q) \in (\mathbb{R}^+)^q$$

$$\gamma_k = (\gamma_{k1}, \ldots, \gamma_{kq}),\ \gamma_{kj} \quad \text{nonnegative integers,}$$

and

$$\overline{Z}(\alpha) = \mathrm{closure}\{\mathrm{Re}\,\lambda : \det(I - \sum A_k e^{-\gamma_k \cdot \alpha\lambda}) = 0\},$$

where $\gamma_k \cdot \alpha = \sum_{j=1}^{q} \gamma_{kj} x_j$. It is shown in [1] that $\overline{Z}(\alpha)$ is a finite union of closed intervals for any $\alpha \in (\mathbb{R}^+)^q$. Also, if the components of α are rationally independents, then $\rho \in \overline{Z}(\alpha)$ if and only if there is a $\theta \in \mathbb{R}^q$ such that

$$\text{(2.3)} \quad H(\rho, \theta, \alpha) = \det[\, I - \sum A_k e^{-\gamma_k \cdot \alpha \rho} e^{i\gamma_k \cdot \theta}\,] = 0.$$

Even if the components of α are not rationally independent, we can still discuss the solutions of (2.3). Let

$$\sigma_1(\alpha) = \max\{\rho \le 0: \text{ there is a } \theta \in \mathbb{R}^q \text{ such that } H(\rho,\theta,\alpha) = 0\}$$
$$\sigma_2(\alpha) = \min\{\rho \ge 0: \text{ there is a } \theta \in \mathbb{R}^q \text{ such that } H(\rho,\theta,\alpha) = 0\}.$$

Take $\sigma_1(\alpha) = -\infty$, $\sigma_2(\alpha) = +\infty$ if the set involved is empty.

From the results in [1], Hypothesis (H1) is equivalent to

$$\sigma_1(\alpha) < -\delta < 0 < \delta < \sigma_2(\sigma) \quad \text{for} \quad \alpha \in V_{\alpha_0}.$$

But then, from [1], $D^0(\alpha)\phi$ is hyperbolic for <u>any</u> $\alpha \in (\mathbb{R}^+)^q$, that is, the property of being locally hyperbolic in the delays is, in fact, a global property in the delays.

A trivial example is $D^0(\alpha)\phi = \phi(0) - a\phi(-\alpha_1) - b\phi(-\alpha_2) + ab\phi(-\alpha_1-\alpha_2)$ with $|a| < 1$, $|b| > 1$. In this case, $H^0(\lambda) = (1-ae^{-\lambda\alpha_1})(1-be^{-\lambda\alpha_2})$ with roots lying on either the line $\mathrm{Re}\,\lambda = \frac{1}{\alpha_1}\ln|a|$ or $\mathrm{Re}\,\lambda = \frac{1}{\alpha_2}\ln|b|$.

The scalar two-delays functional

$$D^0(\alpha)\phi = \phi(0) - a_1\phi(-\alpha_1) - a_2\phi(-\alpha_2)$$

is hyperbolic for any $\alpha \in (\mathbb{R}^+)^2$ if and only if $|a_1| + |a_2| < 1$ (stable) or $|a_2| > 1 + |a_1|$ or $|a_1| > 1 + |a_2|$ (unstable).

Our second hypothesis is the following:

(H2) The characteristic matrix

$$H(\alpha,\lambda) = A_0(\alpha) - \sum_k A_k(\alpha)e^{-\lambda r_k} - \int_{-r}^{0} A(\alpha,\theta)e^{\lambda\theta}d\theta, \quad \lambda \in ¢,$$

is continuously differentiable in α; the characteristic equation $\det H(\alpha,\lambda) = 0$ has, for $\alpha = \alpha_0$, a simple purely imaginary root $\lambda_0 = i\nu_0$, $\nu_0 > 0$, and for any integer $n \neq \pm 1$, $n\lambda_0$ is not a root.

By changing the time scale, we can suppose that $\lambda_0 = i$. By the I.F.T., we can find a $\delta > 0$ and a function $\lambda(\alpha) \in ¢$ continuously differentiable for $|\alpha-\alpha_0| < \delta$ such that $\lambda(\alpha_0) = \lambda_0$, $\lambda(\alpha)$ is a simple root of $\det H(\alpha,\lambda) = 0$ and if $\det H(\alpha,\lambda) = 0$ for $|\alpha-\alpha_0| < \delta$, $|\lambda-\lambda_0| < \delta$, then $\lambda = \lambda(\alpha)$.

Let us take $v(\alpha) \in ¢^n$ and $w(\alpha) \in (¢^n)^*$ such that $H(\alpha,\lambda(\alpha))\, v(\alpha) \equiv 0$, $w(\alpha)H(\alpha,\lambda(\alpha)) \equiv 0$, $|w(\alpha)| = 1$, and

$$w(\alpha) \frac{\partial}{\partial\lambda} H(\alpha,\lambda(\alpha))v(\alpha) \equiv 1 \tag{2.4}$$

for $|\alpha-\alpha_0| < \delta$.

We observe that the derivative of $\lambda(\alpha)$ with respect to α at $\alpha = \alpha_0$ is given by

$$\lambda'(\alpha_0)\cdot\Delta\alpha = -w[\frac{\partial H}{\partial\alpha}(\alpha_0,\lambda_0)\Delta\alpha]v \tag{2.5}$$

where $v = v(\alpha_0)$ and $w(\alpha_0) = w$, for all $\Delta\alpha \in \mathscr{A}$.

Our third hypothesis concerns the differentiability of the Equation (2.1) with respect to the parameter α. If we impose that D and g are continuously differentiable in α, the case where α contains some of the delays r_k is not included, since in this case the function $\alpha \to D(\alpha) \in \mathscr{L}(C,\mathbb{R}^n)$ is not even continuous. Fortunately, all we need is

(H3) For any $K > 0$, any $\phi \in C$ with $\dot{\phi} \in C$, $|\dot{\phi}| \leq K$, the functions

$$\alpha \to D(\alpha,\phi)$$
$$\alpha \to g(\alpha,\phi),$$

are continuously differentiable in V_{α_0}.

The last hypothesis says that the characteristic root $\lambda(\alpha)$ "crosses" the imaginary axis through λ_0, with nonvertical velocity, for almost all directions:

(H4) $$\mathrm{Re}\, \frac{\partial\lambda}{\partial\alpha}(\alpha_0) \neq 0.$$

We can now state an extension of the Hopf Bifurcation Theorem for functional equations:

<u>Theorem</u>. Under Hypotheses [H1]-[H4], there is an $\varepsilon > 0$ such that for $a \in \mathbb{R}$, $|a| < \varepsilon$, there is a C^1-manifold $\Gamma_a \subset \mathscr{A}$, of codimension 1, Γ_a continuously differentiable in a, $\alpha_0 \in \Gamma_0 = \{\alpha \in \mathscr{A}: \mathrm{Re}\,\lambda(\alpha) = 0, |\alpha-\alpha_0| < \varepsilon\}$, such that for every $\alpha \in \Gamma_a$, there is a function $\omega(a,\alpha)$

and an $\omega(a,\alpha)$-periodic function $x^*(a,\alpha)(t)$, continuous together with their first derivatives in t,a,α, $\omega(0,\alpha_0) = \frac{2\pi}{\nu_0}$, $x^*(0,\alpha_0) = 0$, and $x^*(a,\alpha)$ is a solution of Equation (2.1). Furthermore, for $|\alpha-\alpha_0| < \varepsilon$, $|\omega-\omega_0| < \varepsilon$ every ω-periodic solution x of Equation (2.1) with $|x| < \varepsilon$ must be of the above type except for a translation in phase, that is, there exist $a \in (-\varepsilon,\varepsilon)$, $\alpha \in \Gamma_a$ and $b \in \mathbb{R}$ such that $x(t) = x^*(t+b,a,\alpha)$ for all $t \in \mathbb{R}$.

Proof. The proof proceeds as the proof of Theorem 2.1 in [4].

We first introduce a free parameter β in Equation (2.1) by scaling the time and determine 2π-periodic solutions of the resulting equation.

Let $\beta > -1$, $t = (1+\beta)\tau$, $u(\tau) = x((1+\beta)\tau)$ and $u_{\tau,\beta}(\theta) = u(\tau + \frac{\theta}{1+\beta})$, $-r \leq \theta \leq 0$. Then, Equation (2.1) is equivalent to the equation

$$D(\alpha)u_{\tau,\beta} - g(\alpha,u_{\tau,\beta}) = 0. \tag{2.6}$$

If this equation has a 2π-periodic solution, then Equation (2.1) has a $(1+\beta)2\pi$-periodic solution, and conversely.

Let us consider the above equation as a perturbation of the linear equation $D(\alpha_0)u_\tau = 0$ and rewrite it in the form

$$D(\alpha_0)u_\tau = N(\beta,\alpha,u)(\tau)$$

where

$$N(\beta,\alpha,u)(\tau) =_{def} D(\alpha_0)u_\tau - D(\alpha)u_{\tau,\beta} + g(\alpha,u_{\tau,\beta})$$

for all $\beta > -1$, $\alpha \in A$, $u \in \mathcal{P}_{2\pi}$, $\tau \in \mathbb{R}$.

Our program now is to find all the 2π-periodic continuous solutions, which are near to the null solution, of the equation

$$(2.7) \qquad D(\alpha_0)u_\tau = [(I-J)N(\beta,\alpha,u)](\tau),$$

and prove that there exist values of β and α such that $JN(\beta,\alpha,u)(\tau) \equiv 0$. Here J is the projection operator given by relation (1.4).

It is clear that, after this is done, we have a 2π-periodic solution of Equation (2.6).

By Lemma 1.5, Equation (2.7) is equivalent to

$$(2.8) \qquad u(\tau) = a(e^{i(\tau+b)}v+e^{-i(\tau+b)}\bar{v}) + \mathcal{K}(I-J)N(\beta,\alpha,u)(\tau)$$

where a,b are real constants, v is given in (1.3) and the operator $\mathcal{K}$ is given in (1.5).

Since Equation (2.6) is autonomous and equivalent to Equation (2.8) plus $JN(\beta,\alpha,u) = 0$, we can take $b = 0$ in Equation (2.8); the other solutions are obtained by translations in the phase.

Consider the mapping $\mathcal{F}: \mathbb{R} \times (-1,\infty) \times \mathcal{A} \times \mathcal{P}_{2\pi} \to \mathcal{P}_{2\pi}$ defined by

$$\mathcal{F}(a,\beta,\alpha,u)(\tau) = u(\tau) - a(e^{i\tau}v+e^{-i\tau}\bar{v}) - \mathcal{K}(I-J)N(\beta,\alpha,u)(\tau).$$

It is easy to see that $\frac{\partial\mathcal{F}}{\partial a}$, and $\frac{\partial\mathcal{F}}{\partial x}$ exist and are continuous functions in all the arguments.

We note that $\mathcal{F}(0,0,\alpha_0,0) = 0$ and $\frac{\partial \mathcal{F}}{\partial u}(0,0,\alpha_0,0)$ is the identity.

Then, the I.F.T. implies the existence of an $\varepsilon > 0$ and a function $u^*(a,\beta,\alpha) \in \mathcal{P}_\omega$, for each (a,β,α) such that $|a| < \varepsilon$, $|\beta| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon$, $|u^*(a,\beta,\alpha)| < \varepsilon$, $u^*(0,\beta,\alpha) = 0$, and $\mathcal{F}(a,\beta,\alpha,u^*(a,\beta,\alpha)) = 0$.

As a consequence of the I.F.T., $u^*(\tau,a,\beta,\alpha)$ is continuously differentiable in a for $|a| < \varepsilon$. We want to show that it is also continuously differentiable in τ, β and α in the region $\tau \in \mathbb{R}$, $|\beta| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon$. The continuous differentiability with respect to τ follows from the fact that $u^*(a,\beta,\alpha)(\tau)$, being a periodic solution of the functional equation (2.7), is continuously differentiable with respect to τ, by Corollary 1.4. Now, $\frac{d}{d\tau} u^*$ continuous and the fact that $N(\beta,\alpha,u)$ is continuously differentiable in β whenever u is continuously differentiable in τ, implies $u^*(a,\beta,\alpha)$ is continuously differentiable in β. By the same reason and Hypothesis (H4), $u^*(a,\beta,\alpha)$ is continuously differentiable with respect to α.

Let us consider now the bifurcation equations $J(N(\beta,\alpha,u^*(a,\beta,\alpha))) = 0$ or, equivalently the equation,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau_w} N(\beta,\alpha,u^*(a,\beta,\alpha))(\tau)d\tau = 0$$

for $|a| < \varepsilon$, $|\beta| < \varepsilon$, $|\alpha-\alpha_0| < \varepsilon$. Since $J[N(\beta,\alpha,u^*(0,\beta,\alpha))] = 0$ for all α and β, we divide this last integral by a and define

$$G(a,\beta,\alpha) = \frac{1}{a}\cdot\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau_w} N(\beta,\alpha,u^*(a,\beta,\alpha))(\tau)d\tau$$

and solve $G(a,\beta,\alpha) = 0$.

Now, we compute $G(0,\beta,\alpha)$. In order to simplify the notation, let us put $u^1(\tau,\beta,\alpha) =_{\text{def}} \frac{\partial}{\partial a} u^*(\tau,a,\beta,\alpha)|_{a=0}$. Then, we have, since $g'(\alpha,0) = 0$ for all α:

$$G(0,\beta,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\tau} w[D(\alpha_0)u^1_\tau(\beta,\alpha) - D(\alpha)u^1_{\tau,\beta}(\beta,\alpha)]d\tau,$$

$$= w[I - \int_{-r}^{0} d\mu(\alpha_0,\theta)e^{i\theta}]\frac{1}{2\pi}\int_0^{2\pi} e^{-is}u^1(s,\beta,\alpha)ds -$$

$$- w[I - \int_{-r}^{0} d\mu(\alpha,\theta)e^{i\frac{\theta}{1+\beta}}]\frac{1}{2\pi}\int_0^{2\pi} e^{-is}u^1(s,\beta,\alpha)ds]$$

$$= w[H(\alpha_0,i) - H(\alpha,\tfrac{i}{1+\beta})]v = -wH(\alpha,\tfrac{i}{1+\beta})v.$$

Therefore, taking derivatives and using the equations $w\frac{\partial H}{\partial \lambda}(\alpha_0,i)v = 1$ and $\lambda'(\alpha_0)\Delta\alpha = -w[\frac{\partial H}{\partial \alpha}(\alpha_0,i)\Delta\alpha]v$ we get

$$\frac{\partial G}{\partial \beta}(0,0,\alpha_0) = i \tag{2.9}$$

and

$$\frac{\partial G}{\partial \alpha}(0,0,\alpha_0)\cdot\Delta\alpha = -w\frac{\partial H}{\partial \alpha}(\alpha_0,i)\Delta\alpha\cdot v = \lambda'(\alpha_0)\cdot\Delta\alpha \tag{2.10}$$

From (2.9) and the I.F.T., we can solve the equation

$$\operatorname{Im} G(a,\beta,\alpha) = 0$$

for $\beta = \beta^*(a,\alpha)$ about $a = 0$, $\alpha = \alpha_0$.

Consider now the last equation

$$\Gamma(a,\alpha) =_{\text{def}} \text{Re } G(a,\beta^*(a,\alpha),\alpha) = 0.$$

Relation (2.10) and Hypothesis (H4) implies that for each $a \in \mathbb{R}$, $|a|$ small, there exists a C^1-manifold $\Gamma_a \subset \mathscr{A}$, with codimension 1, such that for any $\alpha \in \Gamma_a$, $\Gamma(a,\alpha) = 0$. Moreover,

$$\Gamma_0 = \{\alpha \in \mathscr{A}: |\alpha-\alpha_0| < \varepsilon, \text{ Re } wH(\alpha, \frac{i}{1+\beta^*(0,\alpha)})v = 0\}$$
$$= \{\alpha \in \mathscr{A}: |\alpha-\alpha_0| < \varepsilon, \text{ Re } \lambda(\alpha) = 0\}$$

and the tangent space of Γ_0 at α_0 is given by $T_{\alpha_0}\Gamma_0 =$ Ker Re $\lambda'(\alpha_0)$.

Given an $a \in \mathbb{R}$, $|a| < \varepsilon$, and an $\alpha \in \Gamma_a$, we determine $\beta^*(a,\alpha)$ and find a nonconstant 2π-periodic solution $u^*(\tau,a,\beta^*(a,\alpha),\alpha)$ of Equation (2.6) and so, $x^*(t,a,\alpha) = u^*(\frac{t}{1+\beta^*(a,\alpha)},a,\beta^*(a,\alpha),\alpha)$ is a $[1+\beta^*(a,\alpha)]\cdot 2\pi$-periodic solution of Equation (2.1).

This completes the proof of the theorem.

Remark. It is clear from the proof of the above results that if we assume that the functions $D(\alpha,\phi)$ and $g(\alpha,\phi)$ have k derivatives with respect to ϕ which are continuous in the pair (α,ϕ) and Hypothesis (H3) is satisfied for the k^{th} derivative with respect to α when ϕ has k continuous derivatives in $\theta \in [-r,0]$, then, the bifurcation function $\Gamma(a,\alpha)$ and the functions $x(a,\alpha)$ and

$\omega(a,\alpha)$ in the theorem have continuous derivatives up through order k.

3. Example.

Consider the scalar equation

$$(3.1) \qquad x(t) = \int_{-L_1-L_2}^{-L_1} F(x(t+\theta))\,d\theta$$

where $F(x) = bx(1-x)$ if $x \in [0,1]$, $F(x) = 0$ if $x \notin [0,1]$, and $b > 0$, $L_1 \geq 0$ and $L_2 > 0$ are constants, $bL_2 > 1$.

Equation (3.1) was proposed in [2] as a model for some epidemics and growth processes.

The dynamical system defined by Equation (3.1) has the equilibrium $1 - (bL_2)^{-1}$. The variational equation around this equilibrium is given by

$$(3.2) \qquad y(t) = \left(\frac{2}{L_2} - b\right) \int_{-L_1-L_2}^{-L_1} y(t+\theta)\,d\theta .$$

Let $\alpha = (b,L_1,L_2)$. The characteristic equation is

$$(3.3) \qquad H(\alpha,\lambda) = 1 - \left(\frac{2}{L_2} - b\right) \int_{-L_1-L_2}^{-L_1} e^{\lambda\theta}\,d\theta .$$

Since $bL_2 > 1$, $H(\alpha,0) \neq 0$ and zero is not a characteristic value. For $\lambda \neq 0$, finding the zeros of $H(\alpha,\lambda)$ is equivalent to finding the zeros of the function

$$(3.4) \qquad \lambda H(\alpha,\lambda) = \lambda - \left(\frac{2}{L_2} - b\right)\left[e^{-L_1\lambda} - e^{-(L_1+L_2)\lambda}\right].$$

For this example, $D^0\phi = \phi(0)$ which is hyperbolic; in fact it is asymptotically stable since $D^0 y_t = 0$ implies y_t is the zero function for $t \geq L_1 + L_2$ regardless of the initial function. The operator D^1 is given by

$$D^1\phi = (\frac{2}{L_2} - b) \int_{-L_1-L_2}^{-L_1} \phi(\theta)d\theta$$

which satisfies all of the smoothness conditions required in the Hopf bifurcation theorem. Also, the function $F(x)$ is smooth near the equilibrium point $1 - (bL_2)^{-1}$.

It remains to verify the hypotheses (H2), (H4) concerning the nontrivial zeros of the function $\lambda H(\alpha,\lambda)$. If $\beta = \frac{2}{L_2} - b$, this is equivalent to discussing the zeros of

$$\lambda e^{(L_1+L_2)\lambda} = \beta(e^{L_2\lambda} - 1). \tag{3.5}$$

The most interesting values of β, L_1, L_2 are those for which all nontrivial solutions have real parts less than or equal to zero and there are roots with real part equal to zero. This will define a surface Γ of codimension one in $\mathbb{R}^3$. On one side of this surface, the equilibrium point $1 - (bL_2)^{-1}$ will be asymptotically stable and, on the other it will be unstable. If a point of this surface corresponds to the case where there are only two purely imaginary roots, then we have a Hopf bifurcation.

The analysis of the characteristic equation and the determination of the surface Γ are difficult. However, we can say something about the equation.

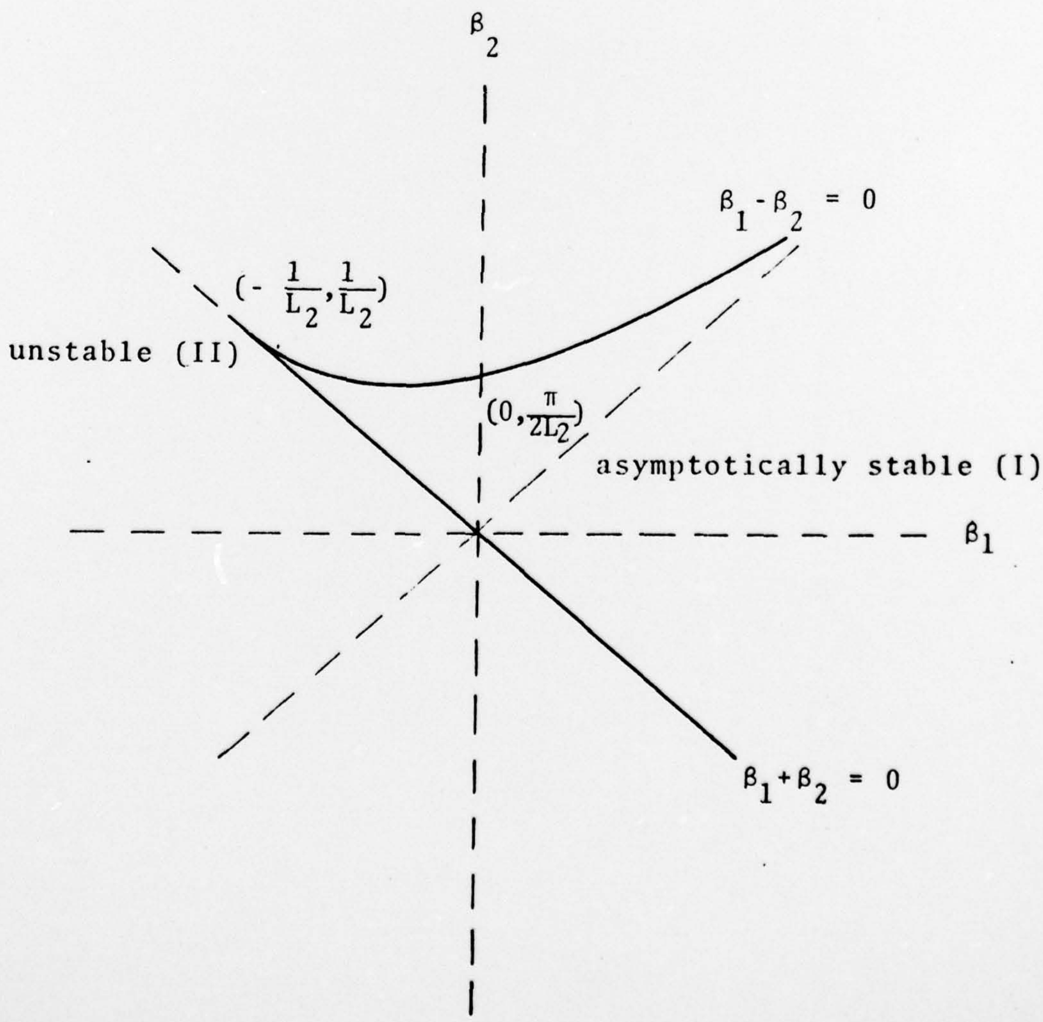
Suppose first that $L_1 = 0$, that is,

$$\lambda e^{L_2\lambda} = \beta(e^{L_2\lambda} - 1). \tag{3.6}$$

The region of stability and instability for this equation are well understood (see [3]). In fact, for the more general equation

$$\lambda e^{L_2\lambda} = -\beta_1 e^{L_2\lambda} - \beta_2 \tag{3.7}$$

the regions are depicted in the accompanying figure.



In region (I), all characteristic roots have negative real parts. The curve $\beta_1 + \beta_2 = 0$ bounding this region corresponds to a simple zero root and the other curve bounding this region, to two purely imaginary roots. At the intersection of these curves one has a double zero root.

The roots of (3.6) of interest to us are the nonzero ones since they must be roots of (3.3) and we have assumed $1 - bL_2 \neq 0$. From our discussion of (3.7), it follows that Equation (3.6) has all roots with negative real parts for $1 - (bL_2)^{-1} > 0$, which is satisfied by our hypothesis that $bL_2 > 1$. Consequently, the equilibrium point $1 - (bL_2)^{-1}$ is asymptotically stable for all b, L_2 and no bifurcation can occur.

Now suppose $L_1 > 0$ and for simplicity that $\beta \leq 0$. Equation (3.5) has only one real root, namely, the root $\lambda = 0$. This means Equation (3.3) has no real roots. Let us prove that, if L_1 and L_2 vary in a compact interval, then as $\beta \to 0$, $\beta < 0$, all the nonreal roots have real part approaching $-\infty$. It is easy to see that $\lambda = 0$ is a simple root of the function $\lambda - \beta[e^{-L_1\lambda} - e^{-(L_1+L_2)\lambda}]$, for all β, L_1, L_2. Therefore, by the I.F.T. we can find an $\varepsilon > 0$ such that for $|\beta| < \frac{\varepsilon^2}{4}$ there is a unique root in the ball $|\lambda| < \varepsilon$, namely $\lambda = 0$. By equating the real and imaginary parts, we see that any root λ satisfies $\mathrm{Re}\,\lambda \leq 2|\beta|$ and if $\mathrm{Re}\,\lambda > \delta \ln|\beta|$, $\delta > 0$, then $|\mathrm{Im}\,\lambda| \leq 2|\beta|^{1-\delta(L_1+L_2)}$. Choose any $\delta > 0$ such that $\delta < [2(L_1+L_2)]^{-1}$. Then, each root in the rectangle $\{\lambda \in \mathbb{C}: \delta \ln|\beta| \leq \mathrm{Re}\,\lambda \leq 2|\beta|, |\mathrm{Im}\,\lambda| \leq 2|\beta|^{1/2}\}$ satisfies $|\lambda| \leq 2|\beta|^{1/2}$. Therefore for $|\beta|$ small all roots, except $\lambda = 0$, have real parts less than $\delta \ln|\beta|$. If $\lambda = i\omega$ is a solution of Equation (3.5), then

$$\cos L_1\omega = \cos(L_1+L_2)\omega$$

(3.8)

$$\beta = \frac{-\omega}{\sin L_1\omega - \sin(L_1+L_2)\omega}$$

which implies that

(3.9) $$\beta \sin \frac{2k\pi L_1}{2L_1+L_2} = -\frac{k\pi}{2L_1+L_2}, \quad \omega = \frac{2k\pi}{2L_1+L_2}$$

for some integer $k \geq 1$.

Equation (3.9) define surfaces in (β,L_1,L_2)-space. These are certainly surfaces which always have $\beta < 0$. Take the surface S that has the following property: For any fixed (β^0,L_1^0,L_2^0) in S, Equation (3.5) at (β,L_1^0,L_2^0) has all nonzero eigenvalues with negative real parts for $\beta^0 < \beta < 0$. At the point (β^0,L_1^0,L_2^0), there are exactly two purely imaginary roots of Equation 3.5. Thus, the condition (H2) is satisfied. Also, in the direction normal to this surface the eigenvalues corresponding to these two purely imaginary eigenvalues are crossing with a definite velocity. Therefore, Hypothesis (H4) is satisfied and there is a Hopf bifurcation at each point on the surface.

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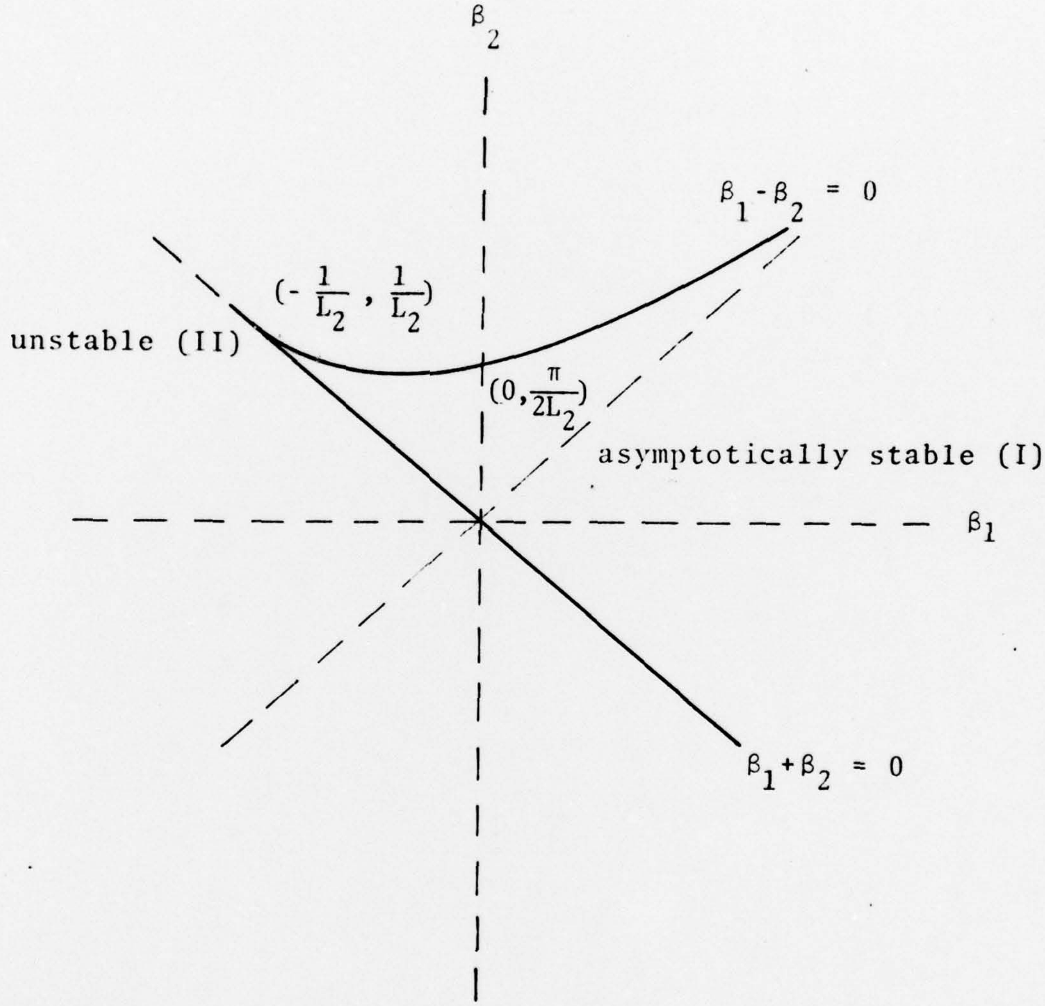

β_2
$\beta_1 - \beta_2 = 0$
$(-\frac{1}{L_2}, \frac{1}{L_2})$
unstable (II)
$(0, \frac{\pi}{2L_2})$
asymptotically stable (I)
β_1
$\beta_1 + \beta_2 = 0$